CHAPTER

TWELVE

APPROXIMATION THEORY

12-1 THE APPROXIMATION PROBLEM

In all previous chapters, it was tacitly assumed that a realizable network function (driving-point or transfer) was given at the outset of the design procedure, presumably as part of the specifications. In practice, of course, this is almost never the case; normally, the desired circuit performance is prescribed either in the form of a chart or a graph or, more often, as a set of inequalities. A typical set of specifications may be in the form given in the following example.

Example 12-1 Design a doubly terminated reactance two-port with both terminations equal to 50 Ω and with the following restrictions on the transducer loss α:

α : loss in dB

$$\alpha \leq 1 \text{ dB} \quad \text{for} \quad 0 \leq f \leq 1.8 \text{ MHz}$$
$$\alpha \geq 50 \text{ dB} \quad \text{for} \quad 7 \text{ MHz} \leq f \leq \infty$$

G: gain in dB

G=-α

The solution of this problem will be discussed later in the chapter. Here, we merely note that the specified circuit is evidently intended to pass low frequencies (below 1.8 MHz) essentially unattenuated and suppress high frequencies (above 7 MHz) strongly.

Such frequency-selective two-ports are called *filters*. Depending on the frequency bands passed (or suppressed), they can be classified as low-pass, or high-pass, bandpass, or bandstop, etc., filters. The circuit specified above is clearly a low-pass filter. The frequency region in which the loss must be low ($0 \le f \le 1.8$ MHz in the above example) is called the *passband* of the filter; the high-loss region (7 MHz $\le f \le \infty$ in the example) is called the *stopband*. The frequency limits of these bands are called *passband limit* and *stopband limit*, respectively. In the above example, the passband limit is $f_p = 1.8$ MHz, and the stopband limit is $f_s = 7$ MHz.

The branch of circuit theory which deals with finding a realizable network function meeting such practical specifications is called *approximation theory*. Its fundamentals will be discussed in this chapter.

It will be demonstrated next that the approximation problem is especially

$$\alpha = 10\log_{10}(1+|K^2|)$$

$|K^2|$: characteristic function

$$K(s) \rightarrow K(j\omega)$$

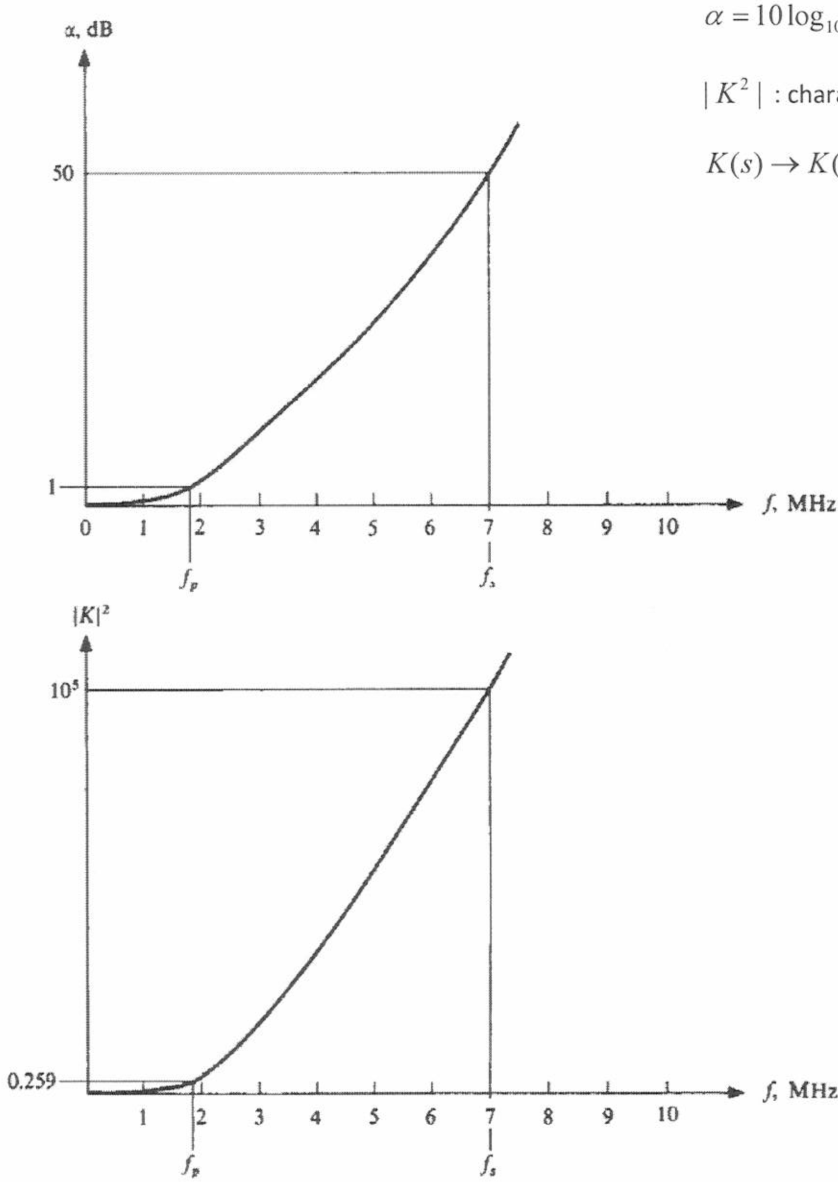


Figure 12-1 Loss and $|K|^2$ responses for a low-pass filter.

conveniently solved in terms of the transducer function $H(s)$ and the characteristic function $K(s)$ [introduced in Chap. 6 via Eqs. (6-5) and (6-21), respectively]. Specifically, if only the behavior of the transducer loss $\alpha(\omega)$ is specified, it is advantageous to find $|K(j\omega)|^2$; if the phase or time response is also prescribed, $H(s)$ should be calculated. The reason for these preferences is that, by Eqs. (6-8) and (6-25), the transducer loss for $s = j\omega$ is

$$\alpha = 10 \log |H|^2 = 10 \log (|K|^2 + 1) \tag{12-1}$$

Consequently, whenever $\alpha \approx 0$, $|K|^2 \approx 0$; and whenever $\alpha \to \infty$, $|K|^2 \to \infty$. Similarly, if $\alpha(\omega)$ is increasing (decreasing), $|K(j\omega)|^2$ must also be increasing (decreasing). In conclusion, the loss $\alpha(\omega)$ and the squared modulus of $K(j\omega)$ vary exactly the same way with frequency, except that there is of course a difference between their actual numerical values. This point is illustrated in Fig. 12-1, which compares the two responses for the low-pass filter specified in the above example. Obviously, neither curve is drawn to a true scale.

Another advantage of using $|K|^2$ for approximating a prescribed filter loss response is based on the fact (to be demonstrated) that it is efficient to place all zeros of the $\alpha(\omega)$ function in those frequency bands where α must be small, i.e., in the *passbands*, and, similarly, that all poles of $\alpha(\omega)$ should be in the frequency bands of specified high attenuation, i.e., in the *stopbands*. By Eq. (12-1), the zeros and poles of $|K|^2$ are located at the same frequencies as those of α. Hence, the approximate locations of the zeros and poles of $|K|^2$ are usually known in advance, and these poles and zeros are normally all on the $j\omega$ axis of the s plane. By contrast, the zeros of $H(s)$ are in the inside of the left half s-plane. Finding the zeros of $H(s)$ is therefore a two-dimensional problem, while the search for those of $K(s)$ is usually restricted to the $j\omega$ axis, i.e., is in one dimension only.

Example 12-2 To illustrate how easily a filter-type response can be obtained by properly choosing the location of the zeros and poles of $|K|^2$, assume that we wish α to be small for $0 \le \omega \le 1$ and as large as possible for $2 \le \omega \le 4$. Let $K(s)$ be of degree 4; then in general

$$K(s) = C\frac{(s - s_1)(s - s_2)(s - s_3)(s - s_4)}{(s - p_1)(s - p_2)(s - p_3)(s - p_4)}$$

Since $K(s)$ is a *real* rational function, the s_i and p_i must occur in conjugate pairs. Furthermore, we anticipate that distributing the *zeros* s_i along the $j\omega$ axis in the $0 \le \omega \le 1$ range, the values of $|K|$ and α will be small in that range. Finally, placing the *poles* p_i on the $j\omega$ axis in the $2 \le \omega \le 4$ range should make $|K|$ and hence also α large in that frequency range. Hence, choosing the zeros and poles arbitrarily at the plausible locations given by

$$s_1 = s_2^* = j0.3 \qquad s_3 = s_4^* = j0.7 \qquad p_1 = p_2^* = j2.5 \qquad p_3 = p_4^* = j3.5$$

and selecting $C = 100$ gives the resulting $K(j\omega)$ in the form

$$K(j\omega) = 100\frac{(\omega^2 - 0.3^2)(\omega^2 - 0.7^2)}{(\omega^2 - 2.5^2)(\omega^2 - 3.5^2)}$$

$$\frac{1}{S_{21}} = H(s) = \frac{1}{2}\sqrt{\frac{R_L}{R_G}}\,\frac{E_G}{V_2(s)}$$

Characteristic function:

$$K(s) = \rho_1(s)H(s)$$
$$s \to j\omega$$
$$|H|^2 = \frac{P_{\max,gen}}{P_{out}}$$
$$|K|^2 = \frac{P_{refl}}{P_{out}} = \frac{P_{\max} - P_{in}}{P_{out}}$$
$$|H|^2 = |K|^2 + 1$$
$$Gain(dB) = G(f)$$
$$-\alpha = -10\log|H|^2$$

For stability, the zeros of H(s) (natural modes) must be inside the LHP of the s plane (including zero at $s \to \infty$)

Feldtkeller

$H(s)H(-s) = K(s)K(-s) + 1$ ~~Tedtkeller~~ equation

H(s) : real rational function of s $\to \dfrac{1}{S_{21}(s)}$

K(s) : real rational function of s

$\rho_{\substack{1\\2}}(s)$: real rational function of s $\to -s_{11}, s_{22}$

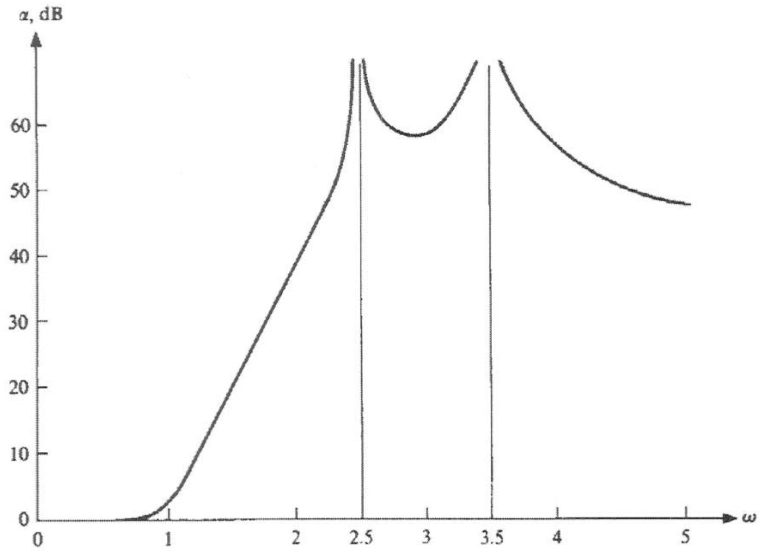


Figure 12-2 Loss response for a filter with $j\omega$-axis zeros and poles.

The loss response $\alpha(\omega)$ thus obtained is shown in Fig. 12-2. It confirms the effectiveness of distributing the zeros and poles in the passband and stopband, respectively: the passband loss is small (less than 0.1 dB over 80 percent of the passband) and the stopband loss is large.

The situation changes, however, if we are interested in the phase or delay response of our circuit. By Eq. (6-7), the phase lag β between the output and input voltages is simply the phase of the complex function $H(j\omega)$:

$$\beta(\omega) = \tan^{-1}\frac{\text{Im } H(j\omega)}{\text{Re } H(j\omega)} \tag{12-2}$$

$A_v = \frac{V_0}{V_{in}}$

Hence, the phase as well as the *phase delay*

$\beta = \angle H$

$$T_{\text{ph}}(\omega) = \frac{\beta(\omega)}{\omega} \tag{12-3}$$

$-\beta = \angle S_{21}$

and the *group delay*

$$T_g(\omega) = \frac{d\beta(\omega)}{d\omega} \tag{12-4}$$

are closely related to the transducer function $H(s)$ and are easily expressed from it. There is no direct connection between these quantities and $K(s)$. Hence, if $\beta(\omega)$, $T_{\text{ph}}(\omega)$, or $T_g(\omega)$ is specified, it is more expedient to calculate in terms of $H(s)$.

Similarly, the time response of a two-port is given by

$$v_2(t) = \mathscr{L}^{-1}[V_2(s)] = \frac{1}{2}\sqrt{\frac{R_L}{R_G}}\,\mathscr{L}^{-1}\left[\frac{E_G(s)}{H(s)}\right] \tag{12-5}$$

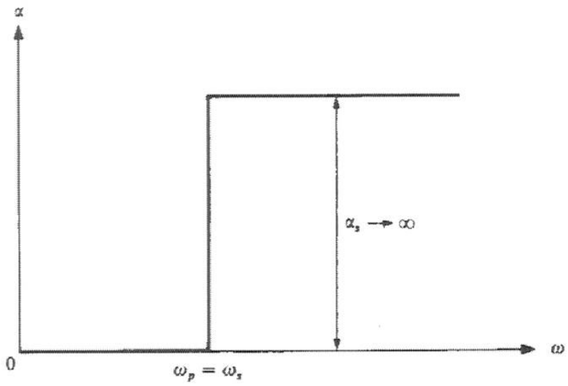


Figure 12-3 Brick-wall low-pass response.

where $\mathscr{L}^{-1}$ denotes the inverse Laplace transform and Eq. (6-5) was utilized. There is no comparable relation in terms of $K(s)$. Therefore, only $H(s)$ is useful for satisfying time-response specifications.

In most design problems, the desired ideal characteristics cannot be exactly achieved. For example, an ideal low-pass filter would have to exhibit the infinitely sharp "brick-wall" loss response illustrated in Fig. 12-3. Such a characteristic is not realizable with any circuit built from a finite number of elements (refer to Probs. 12-1 and 12-2). Hence, the desired loss function must be *approximated* by functions which are realizable. Obviously, we want to achieve the best possible agreement between the desired and the actual performance. Two widely used procedures for measuring this agreement are:

(*a*) To compare the two functions and their first $n-1$ derivatives at one single value of the independent variable.†
(*b*) Or to evaluate the maximum deviation between the two functions in a range of the independent variable.

Criterion *a* requires therefore that

max flat:

a- matches values and derivatives

b- minimizes max error in band

c- minimizes $\int e^p(f)df$ (least - pth)

$$\begin{aligned} F_{\text{spec}}(\omega) &= F_{\text{act}}(\omega) \\ \frac{dF_{\text{spec}}}{d\omega} &= \frac{dF_{\text{act}}}{d\omega} \\ \frac{d^2F_{\text{spec}}}{d\omega^2} &= \frac{d^2F_{\text{act}}}{d\omega^2} \\ &\cdots\cdots\cdots \\ \frac{d^{n-1}F_{\text{spec}}}{d\omega^{n-1}} &= \frac{d^{n-1}F_{\text{act}}}{d\omega^{n-1}} \end{aligned} \tag{12-6}$$

for some $\omega = \omega_0$. Here, $F_{\text{spec}}(\omega)$ is the specified response while $F_{\text{act}}(\omega)$ is the actual one. Obviously, $F_{\text{act}}(\omega)$ must have n free parameters, e.g., coefficients, or

† Usually the frequency ω is the independent variable.

zeros and poles if F_{act} is a rational function of ω, in order to satisfy the n equations in (12-6). If F_{act} does satisfy Eq. (12-6), the error between F_{spec} and F_{act} is called *maximally flat* and F_{act} is a *maximally flat approximation* of F_{spec}.

Criterion b suggests the maximum absolute error in the $\omega_l \leq \omega \leq \omega_u$ range:

$$E = \max_{\omega_l \leq \omega \leq \omega_u} |F_{spec}(\omega) - F_{act}(\omega)| \tag{12-7}$$

be minimized. Assuming again that F_{act} has n adjustable parameters, this will

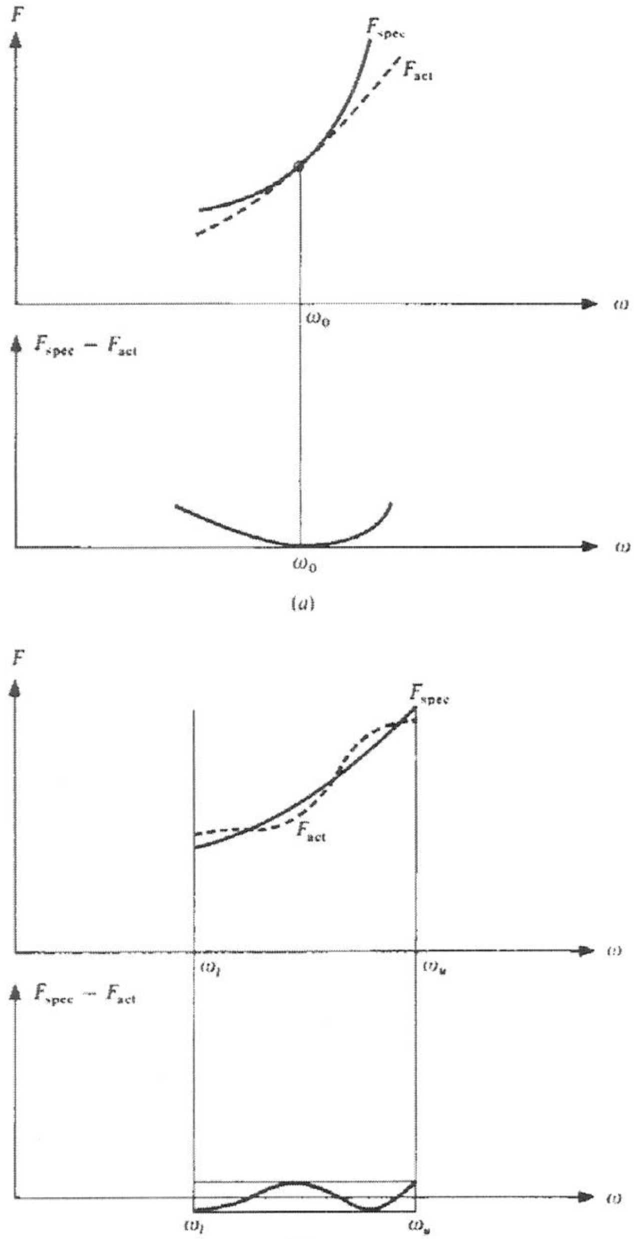


Figure 12-4 (*a*) Maximally flat approximation; (*b*) equal-ripple approximation.

Selectable parameters $p_1, p_2 .... p_n$ ripple size: n+1 equal max errors usually happen if the error $F_{spec}(\omega) - F_{act}(\omega)$ has $n + 1$ equal alternating extrema in the frequency range between ω_l and ω_u. Since such error is called equal-ripple, the approximation itself will be called *equal-ripple* or *min-max approximation.* Responses representing maximally flat and equal-ripple approximations are illustrated in Fig. 12-4*a* and *b*, respectively.

In the following, we shall obtain maximally flat or equal-ripple approximations to some idealized responses, such as the brick-wall response illustrated in Fig. 12-3. From the discussions of Chap. 6 we know that the following information is required for the circuit realization process:

1. The degree n of the circuit, i.e., the number of its natural frequencies
2. The characteristic function $K(s)$ and/or the transducer factor $H(s)$

$K(s)$ and $H(s)$ can be described by the polynomials $E(s)$, $F(s)$, and $P(s)$ or by the critical frequencies, i.e., natural modes, loss poles (transmission zeros), and zero-loss points (reflection zeros) of the circuit.

The next sections will describe the calculation of these functions and parameters for some simple but important responses. Initially, some approximations to the ideal low-pass response of Fig. 12-3 will be discussed. Then it will be shown how transformation of the frequency variable can be used to extend these approximations to the design of high-pass, bandpass, and bandstop filters. After that, the maximally flat approximation of a linear phase, i.e., constant group delay, will be described for low-pass filters. Finally, some simple solutions of the approximation problem for bandpass filters will be discussed.

12-2 BUTTERWORTH APPROXIMATION

One of the simplest techniques for finding a useful realizable function which simulates the brick-wall loss function of Fig. 12-3 is to carry out a maximally flat approximation in the vicinity of $\omega_0 = 0$, where α and all its derivatives vanish. Choosing a *polynomial* rather than rational $K(s)$, the function

$$|K(j\omega)|^2 = K(j\omega)K(-j\omega) \tag{12-8}$$

will also be a polynomial in ω^2:

$$|K(j\omega)|^2 = C_n\omega^{2n} + C_{n-1}\omega^{2(n-1)} + \cdots + C_0 \tag{12-9}$$

Next, observe that α and $|K|^2$ satisfy

$$\begin{aligned} \alpha &= 10 \log (1 + |K|^2) \\ \frac{d\alpha}{d(\omega^2)} &= \frac{10 \log e}{1 + |K|^2} \frac{d|K|^2}{d(\omega^2)} \\ \frac{d^2\alpha}{d(\omega^2)^2} &= \frac{10 \log e}{1 + |K|^2} \left\{ -\frac{1}{1 + |K|^2} \left[\frac{d|K|^2}{d(\omega^2)}\right]^2 + \frac{d^2|K|^2}{d(\omega^2)^2} \right\} \end{aligned} \tag{12-10}$$

. . . .

$$|K(j\omega)|^2 = \left[\mathrm{Re}(K(j\omega))\right]^2 + \left[\mathrm{Im}(K(j\omega))\right]^2$$
$$= \{\mathrm{Re}(K(s))|_{s=j\omega}\}^2 + \{\mathrm{Im}(K(s))|_{s=j\omega}\}^2$$

$$K(s) = a_n s^n + a_{n-1} s^{n-1} + \ldots + a_0$$

$$K(s) = a_n s^n$$

From Eq. (12-10), it follows that whenever the conditions

$$|K|^2 = 0$$
$$\frac{d|K|^2}{d(\omega^2)} = 0 \qquad (12\text{-}11)$$
$$\frac{d^2|K|^2}{d(\omega^2)^2} = 0$$
$$\cdots\cdots\cdots$$

hold for some $\omega = \omega_0$, the same conditions will be valid for α, $d\alpha/d(\omega^2)$, $d^2\alpha/d(\omega^2)^2$, etc., and vice versa. Hence, the maximally flat approximation of the brick-wall response of Fig. 12-3 requires that $|K|^2$ satisfy relations (12-11).

Applying, accordingly, the equations of (12-11) to the function of Eq. (12-9) one by one, we obtain the conditions

$$C_0 = C_1 = C_2 = \cdots C_{n-1} = 0 \qquad (12\text{-}12)$$

The nth derivative of $|K|^2$ can only be matched to zero by setting C_n equal to zero. This, however, would result in $|K|^2 \equiv 0$ and hence $\alpha \equiv 0$ for all ω. Such an allpass network cannot be regarded as a useful approximation of the characteristic of Fig. 12-3. Hence, we keep $C_n > 0$† and obtain, for $s = j\omega$,

$$|K|^2 = C_n \omega^{2n} \qquad K(s) = \pm\sqrt{C_n}\, s^n \qquad (12\text{-}13)$$

This means that by the Feldtkeller equation (6-29)

$$H(s)H(-s) = K(s)K(-s) + 1 = C_n(-1)^n s^{2n} + 1 \qquad (12\text{-}14)$$

Therefore the zeros of $H(s)H(-s)$, that is, the natural modes and their mirror images with respect to the origin, satisfy

$$s_k^{2n} = (-1)^{n-1} C_n^{-1} = \frac{e^{j\pi(n-1+2k)}}{C_n} \qquad k = 1, 2, \ldots, 2n \qquad (12\text{-}15)$$

or

$$s_k = C_n^{-1/2n} e^{j\pi(n-1+2k)/2n} \qquad k = 1, 2, \ldots, 2n \qquad (12\text{-}16)$$

Thus, the s_k lie on a circle in the s plane, with a radius of $C_n^{-1/2n}$, at angles $\pi(n-1+2k)/2n$. The natural modes are, of course, those s_k which lie in the LHP, that is, $s_1, s_2, \ldots, s_n$. Figure 12-5 illustrates the $n = 4$ case.

The approximation represented by Eqs. (12-13) to (12-16) is called (after its first proponent) the *Butterworth approximation*. Filters realized using this process are called *Butterworth filters*.

At this stage, we see from Eq. (12-13) that all reflection zeros are at $s = 0$ and that all loss poles are at $s \to \infty$. We also see from Eq. (12-16) that the natural modes are at

$$s_k = C_n^{-1/2n} e^{j\pi(n-1+2k)/2n} \qquad k = 1, 2, \ldots, n \qquad (12\text{-}17)$$

† $C_n < 0$ would make $|K|^2 < 0$, which is mathematically meaningless.

$$|K|^2 = C_0 + C_1\omega^2 + \ldots + C_n\omega^{2n}$$

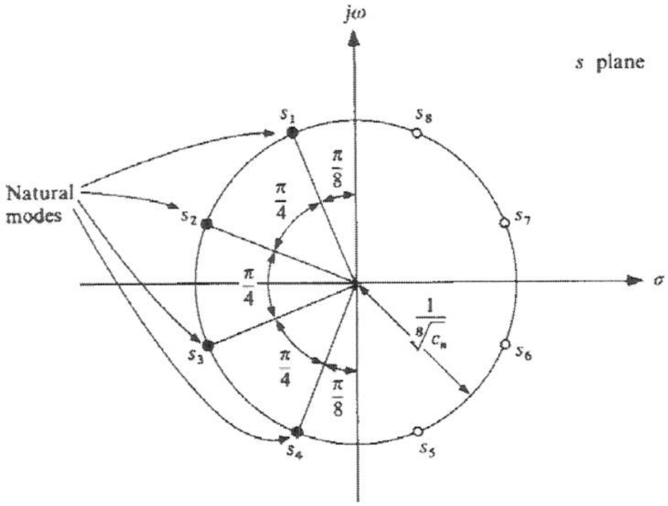
In general: $|K|^2$ vs ω^2 is simple to find. Then:

For $s = j\omega$:

$$|K|^2_{s=j\omega} + 1 = K(j\omega)K(-j\omega) + 1 \Rightarrow K(s)K(-s) + 1$$
$$= |H|^2 \Rightarrow H(s)H(-s)$$

$$\frac{V_{out}(s)}{V_{in}(s)} = \frac{const}{H(s)}$$

zeros of $H(s) \Rightarrow$ natural modes.



$$s_k = \sigma_k + j\omega_k$$
$$e^{s_k t} = e^{\sigma_k t} e^{j\omega_k t}$$

Figure 12-5 Natural modes and their mirror images for a Butterworth filter of degree $n = 4$.

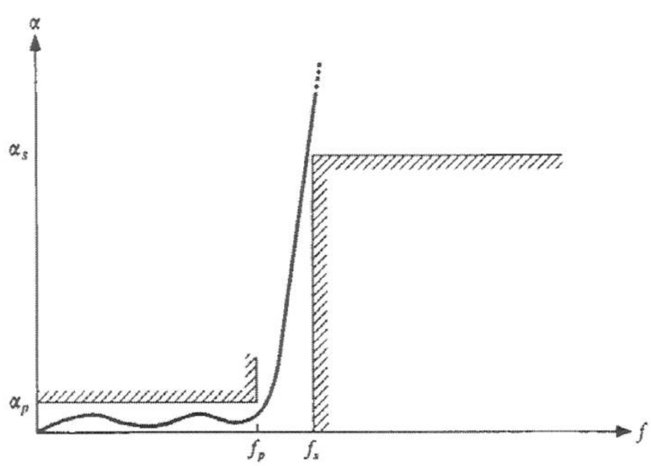
We have yet to find C_n and the necessary degree n of the network. These parameters will depend on the actual specifications for the network. As mentioned earlier, they are usually of the form

$$\begin{aligned} \alpha &\le \alpha_p \quad \text{for } 0 \le f \le f_p \\ \alpha &\ge \alpha_s \quad \text{for } f_s \le f \le \infty \end{aligned} \tag{12-18}$$

where α_p, α_s, f_p, and f_s are prescribed. These conditions can be illustrated graphically, as shown in Fig. 12-6. In order to meet the specifications, the $\alpha(\omega)$ curve must stay *below* the shaded barrier for $0 \le f \le f_p$ and *above* the barrier for $f_s \le f \le \infty$.

Consider now the Butterworth function given in (12-13). By Eq. (12-1),

$$\alpha(\omega) = 10 \log (1 + |K|^2) = 10 \log (1 + C_n \omega^{2n}) \tag{12-19}$$



$$H(s) = K_1 \prod_{i=1}^{n} (s - s_i)$$

$$H(s) = \frac{E}{V_{out}} \to 0$$

$$K(s) = C_n^{1/2} s^n$$
$$\Rightarrow H(s) = K_1 (s - s_1)(s - s_2)....(s - s_n)$$

Figure 12-6 Low-pass filter specifications.

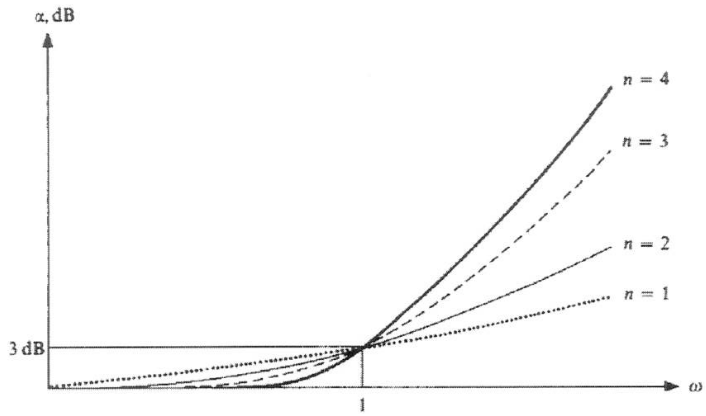


Figure 12-7 Loss responses of Butterworth filters for $n = 1, 2, 3,$ and 4 and $C_n = 1$.

The loss response given by Eq. (12-19) is illustrated in Fig. 12-7 for $C_n = 1$ and for various low values of n. Clearly, the greater n is, the lower the loss in the passband (if we assume $f_p \leq 1$) and the higher in the stopband (for $f_s \geq 1$). Because of the monotonic shape of the $\alpha(\omega)$ curve, if the inequalities (12-18) are satisfied at f_p and at f_s, they must be satisfied for all f. Hence, assuring

$$\alpha(f_p) \leq \alpha_p \qquad \alpha(f_s) \geq \alpha_s \tag{12-20}$$

is sufficient to guarantee that the conditions in Eq. (12-18) are met. Using Eqs. (12-19) and (12-20) gives

$\alpha = 10\log(1 + C_n\omega^{2n})$

$\alpha_p = 10\log(1 + C_n\omega_p^{\ 2n})$

$\alpha_s = 10\log(1 + C_n\omega_s^{\ 2n})$

$$(2\pi f_p)^{2n} \leq \frac{10^{\alpha_p/10} - 1}{C_n} \qquad (2\pi f_s)^{-2n} \leq \frac{C_n}{10^{\alpha_s/10} - 1} \tag{12-21}$$

Since both sides in both inequalities are positive, we can multiply them together to obtain the new inequality

$$\left(\frac{f_p}{f_s}\right)^{2n} \leq \frac{10^{\alpha_p/10} - 1}{10^{\alpha_s/10} - 1} \tag{12-22}$$

At this stage, it is expedient to introduce the *selectivity parameter*

$$k \triangleq \frac{f_p}{f_s} < 1 \tag{12-23}$$

and the *discrimination parameter*

$$k_1 \triangleq \sqrt{\frac{10^{\alpha_p/10} - 1}{10^{\alpha_s/10} - 1}} \approx \frac{\sqrt{0.23\alpha_p}}{10^{\alpha_s/20}} \ll 1 \tag{12-24}$$

As Fig. 12-6 illustrates, the *larger* k is, the more selective, i.e., steeper, the response. Also, the *smaller* k_1 is, the greater the difference between passband and stopband loss.

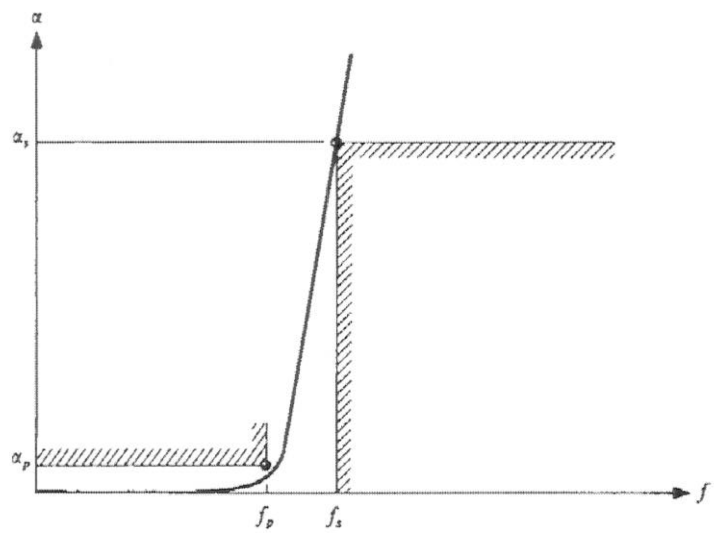


Figure 12-8 Filter response with design margin at f_p.

From Eqs. (12-22) to (12-24), the degree n satisfies

$$n \geq \frac{|\log k_1|}{|\log k|} = \frac{\log (1/k_1)}{\log (1/k)} \tag{12-25}$$

Obviously, in almost all cases the right-hand side of Eq. (12-25) gives a fractional (noninteger) number. Then n should be chosen as the next *higher* integer. Choosing n higher than (12-25) means that the requirements (12-20) will be surpassed. We can, for example, satisfy the specifications at f_s exactly and obtain a safety margin at f_p (Fig. 12-8).† Thus,

$$\alpha(f_s) = \alpha_s \qquad \alpha(f_p) < \alpha_p \tag{12-26}$$

With this choice, from Eq. (12-19),

$$C_n = \frac{10^{\alpha_s/10} - 1}{(2\pi f_s)^{2n}} \tag{12-27}$$

and hence, from Eq. (12-13)

$$|K|^2 = (10^{\alpha_s/10} - 1)\left(\frac{f}{f_s}\right)^{2n} \tag{12-28}$$

Therefore, by (12-1), the loss response is

$$\alpha = 10 \log \left[1 + (10^{\alpha_s/10} - 1)\left(\frac{f}{f_s}\right)^{2n}\right] \tag{12-29}$$

† The response is usually most sensitive to element dissipation, tolerances, etc., around f_p. Hence, we normally try to obtain some design margin there.

Example 12-3 Obtain the necessary degree n, the constant C_n, and the safety margin at f_p for the low-pass filter specified in Sec. 12-1. Find the characteristic function $K(s)$ and the transducer function $H(s)$. Design the circuit.

From Eqs. (12-23) to (12-25),

$$k = \frac{f_p}{f_s} = \frac{1.8}{7} \approx 0.257143$$

$$k_1 = \sqrt{\frac{10^{\alpha_p/10} - 1}{10^{\alpha_s/10} - 1}} = \sqrt{\frac{10^{0.1} - 1}{10^5 - 1}} \approx 1.60912 \times 10^{-3}$$

$$n \geq \frac{-\log k_1}{-\log k} \approx 4.736$$

Hence we choose $n = 5$. If we require (12-26) to hold, then, by (12-27),

$$C_n = \frac{10^{\alpha_s/10} - 1}{(2\pi f_s)^{2n}} = \frac{10^5 - 1}{(2\pi 7 \times 10^6)^{10}} \approx 3.7 \times 10^{-72}$$

This extremely small number and other very small or large numbers which would occur in the remainder of the calculation result from carrying out the calculations without using normalization. To improve the situation, we choose as our frequency unit the half-power (3-dB) radian frequency ω_0 of the response. This frequency is given by

$$|H(j\omega_0)|^2 = \frac{P_{\max}}{P_2} = 2$$

$\Omega = \frac{\omega}{\omega_0}$

$$|K(j\omega_0)|^2 = C_n \omega_0^{2n} = |H(j\omega_0)|^2 - 1 = 1 \qquad (12\text{-}30)$$

$$\omega_0 = C_n^{-1/(2n)}$$

$C_n \to 1$

erefore, in terms of $\Omega = \omega/\omega_0$,

$$|K(j\Omega)|^2 = C_n(\omega_0 \Omega)^{2n} = \Omega^{2n} \qquad (12\text{-}31)$$

A comparison of Eqs. (12-31) and (12-13) shows that the normalization results in replacing C_n by 1. Therefore, if we use Ω and $S = j\Omega$, for this specific example

$$|K|^2 = \Omega^{10} \qquad \text{and} \qquad K(S) = \pm S^5$$

The natural modes are then, by Eq. (12-16), at

$$S_k = e^{j\pi(4+2k)/10} = \cos[\pi(0.4 + 0.2k)] + j\sin[\pi(0.4 + 0.2k)] \qquad k = 1, 2, \ldots, 5$$

Hence

$$H(S) = \pm \prod_{k=1}^{5} (S - S_k) = \pm(S + 1)(S^2 + 0.618034S + 1)(S^2 + 1.618034S + 1)$$

$$= \pm(S^5 + a_4 S^4 + a_3 S^3 + a_2 S^2 + a_1 S + 1)$$

with $a_1 = a_4 = 3.23607$ and $a_2 = a_3 = 5.23607$. If the positive sign in both $K(S)$ and $H(S)$ is chosen and impedance normalization is used so that $R_G = R_L = 1$, Eq. (6-62) gives

$$z_{11} = \frac{H_e - K_e}{H_o + K_o} = \frac{a_4 S^4 + a_2 S^2 + 1}{2S^5 + a_3 S^3 + a_1 S}$$

with the a_k given above. Developing z_{11} into a ladder circuit using the techniques learned in

$$A_v(s) = \frac{1}{\prod_{k=1}^{s}(s - s_k)}$$

"All-pole" transfer function

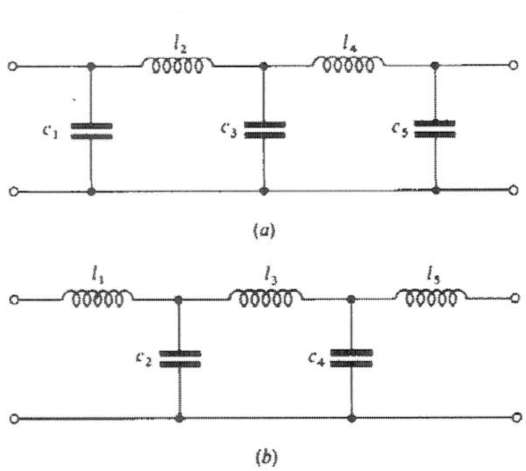


Figure 12-9 Butterworth filters for $n = 5$.

Chaps. 5 and 6, the network of Fig. 12-9*a* results, with the normalized element values $c_1 = c_5 \approx 0.618034$, $c_3 = 2$ and $l_2 = l_4 \approx 1.618034$. The reader should carry out the details as an exercise.

Next, assume that the negative sign is chosen in $K(S)$ so that $K(S) = -S^5$. Then, from Eq. (6-65),

$$y_{11} = \frac{H_e + K_e}{H_o - K_o} = \frac{a_4 S^4 + a_2 S^2 + 1}{2S^5 + a_3 S^3 + a_1 S}$$

Ladder expansion gives now the circuit of Fig. 12-9*b* with $l_1 = l_5 \approx 0.618034$, $l_3 = 2$, and $c_2 = c_4 \approx 1.618034$. Clearly, the two circuits of Fig. 12-9 are duals of each other. Both are normalized in impedance (scaled down by 50) and frequency normalized to the still undetermined 3-dB radian frequency ω_0. To obtain the physical element values, ω_0 must be found. A simple way is provided by Eqs. (12-19) and (12-26), which give

$$\alpha_s = 10 \log \left[1 + \left(\frac{\omega_s}{\omega_0}\right)^{2n}\right]$$

$$\omega_0 = \omega_s (10^{\alpha_s/10} - 1)^{-1/2n} \qquad (12\text{-}32)$$

$$= (2\pi 7 \times 10^6)(10^5 - 1)^{-0.1} \approx 1.39084 \times 10^7 \text{ rad/s}$$

$S = \dfrac{s}{\omega_0}$

$L = \dfrac{L_n}{\omega_0}$

$C = \dfrac{C_n}{\omega_0}$

Hence, the physical-element values can be obtained by multiplying the terminating resistors by $R_0 = 50\ \Omega$, all inductors by $L_0 = R_0/\omega_0 \approx 3.59494\ \mu$H, and all capacitors by $C_0 = 1/R_0\omega_0 \approx 1.43798$ nF. This routine task is left to the reader as an exercise.

We are also often interested in the behavior of the loss response for very small and very large values of ω. For $\Omega \ll 1$, by series expansion,

$$\alpha = 10 \log (1 + \Omega^{2n}) \approx (10 \log e)\, \Omega^{2n} \qquad (12\text{-}33)$$

while for $\Omega \gg 1$

$$\alpha \approx 20n \log \Omega \qquad (12\text{-}34)$$

Table 12-1 Zeros of normalized Butterworth polynomials†

$n = 1$	$n = 2$	$n = 3$	$n = 4$	$n = 5$	$n = 6$	$n = 7$	$n = 8$	$n = 9$	$n = 10$
-1.0000000	-0.7071068 $\pm j0.7071068$	-1.0000000	-0.3826834 $\pm j0.9238795$	-1.0000000	-0.2588190 $\pm j0.9659258$	-1.0000000	-0.1950903 $\pm j0.9807853$	-1.0000000	-0.1564345 $\pm j0.9876883$
		-0.5000000 $\pm j0.8660254$	-0.9238795 $\pm j0.3826834$	-0.3090170 $\pm j0.9510565$	-0.7071068 $\pm j0.7071068$	-0.2225209 $\pm j0.9749279$	-0.5555702 $\pm j0.8314696$	-0.1736482 $\pm j0.9848078$	-0.4539905 $\pm j0.8910065$
				-0.8090170 $\pm j0.5877852$	-0.9659258 $\pm j0.2588190$	-0.6234898 $\pm j0.7818315$	-0.8314696 $\pm j0.5555702$	-0.5000000 $\pm j0.8660254$	-0.7071068 $\pm j0.7071068$
						-0.9009689 $\pm j0.4338837$	-0.9807853 $\pm j0.1950903$	-0.7660444 $\pm j0.6427876$	-0.8910065 $\pm j0.4539905$
								-0.9396926 $\pm j0.3420201$	-0.9876883 $\pm j0.1564345$

† Reproduced by permission from L. Weinberg, "Network Analysis and Synthesis." p. 495, McGraw-Hill, New York, 1962; reprinted by Robert E. Krieger Publishing Co., Inc., Huntington, N.Y., 1975.

Table 12-2 Coefficients of normalized Butterworth polynomials ($a_0 = a_n = 1$ for all n)†

n	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9
1									
2	1.4142136								
3	2.0000000	2.0000000							
4	2.6131259	3.4142136	2.6131259						
5	3.2360680	5.2360680	5.2360680	3.2360680					
6	3.8637033	7.4641016	9.1416202	7.4641016	3.8637033				
7	4.4939592	10.0978347	14.5917939	14.5917939	10.0978347	4.4939592			
8	5.1258309	13.1370712	21.8461510	25.6883559	21.8461510	13.1370712	5.1258309		
9	5.7587705	16.5817187	31.1634375	41.9863857	41.9863857	31.1634375	16.5817187	5.7587705	
10	6.3924532	20.4317291	42.8020611	64.8823963	74.2334292	64.8823963	42.8020611	20.4317291	6.3924532

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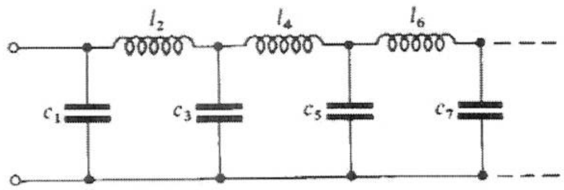
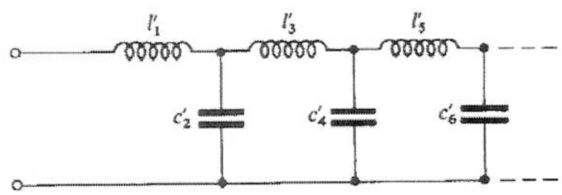
Table 12-3 Element values of Butterworth filters for $R_G = R_L = 1\ \Omega$ and $\omega_0 = 1$ rad/s†

Value of n	c_1 or l'_1	l_2 or c'_2	c_3 or l'_3	l_4 or c'_4	c_5 or l'_5	l_6 or c'_6	c_7 or l'_7	l_8 or c'_8	c_9 or l'_9	l_{10} or c'_{10}
1	2.0000									
2	1.4142	1.4142								
3	1.0000	2.0000	1.0000							
4	0.7654	1.8478	1.8478	0.7654						
5	0.6180	1.6180	2.0000	1.6180	0.6180					
6	0.5176	1.4142	1.9319	1.9319	1.4142	0.5176				
7	0.4450	1.2470	1.8019	2.0000	1.8019	1.2470	0.4450			
8	0.3902	1.1111	1.6629	1.9616	1.9616	1.6629	1.1111	0.3902		
9	0.3473	1.0000	1.5321	1.8794	2.0000	1.8794	1.5321	1.0000	0.3473	
10	0.3129	0.9080	1.4142	1.7820	1.9754	1.9754	1.7820	1.4142	0.9080	0.3129



or



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Hence, doubling Ω results in an increase of

$$\Delta\alpha \approx 6.02n \text{ dB} \tag{12-35}$$

in the value of α. Therefore, on a logarithmic frequency scale, the α–vs.–log Ω curve tends asymptotically to a straight line, with a slope of $6.02n$ dB/octave.†

In the available literature, the normalized roots and coefficients of $H(S)$ are tabulated, as are the element values, for Butterworth filters. Some of these tables are reproduced in Tables 12-1 to 12-3. Our painfully computed circuits (Fig. 12-9) could readily have been found from Table 12-3. The calculation of n and ω_0, however, is necessary even if Table 12-3 is used. As a matter of interest, it should be noted that explicit formulas are also available[1,9] for the element values of Butterworth filters.

† An octave is the interval between two frequencies which have a ratio 1 : 2.

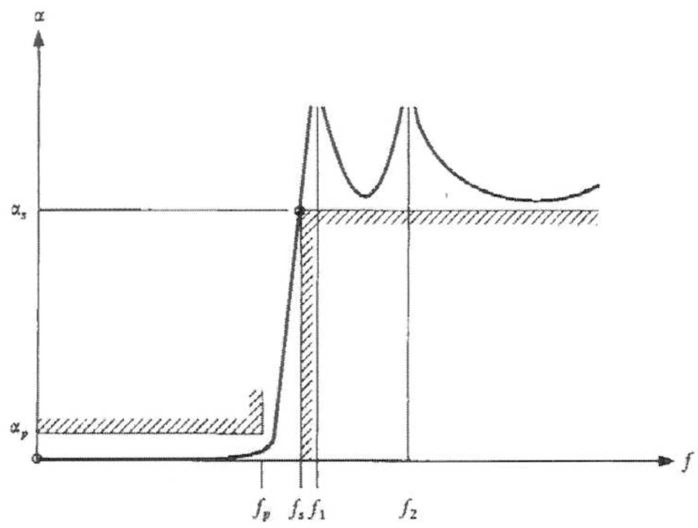

Figure 12-10 Filter loss response with maximally flat passband and finite loss poles.

A minor, but useful, modification of the Butterworth response is obtained if some (or all) of the loss poles are shifted from $\omega \to \infty$ to some finite frequencies ω_i. This results in a rational characteristic function $K(s)$ of the form

Poles at $\pm j\omega$

$$K(s) = \pm\sqrt{C_n}\frac{s^n}{\prod_{i=1}^{k}(s^2+\omega_i^2)} \qquad k \le \frac{n}{2} \tag{12-36}$$

The resulting loss response is still "maximally flat" at the frequency origin; this can easily be verified by repeatedly differentiating the loss function

$$\alpha = 10\log\left[1 + C_n\frac{\omega^{2n}}{\prod_{i=1}^{k}(\omega_i^2-\omega^2)^2}\right] \tag{12-37}$$

The resulting new $\alpha(\omega)$ function is illustrated in Fig. 12-10. It is evident that the selectivity has improved, compared with the response of Fig. 12-8. The price paid is an increase of the number of elements, since the k finite-loss poles require tuned resonant circuits in k of the branches. Also, the previously established design formulas and tables are not applicable to these circuits. Their design can be performed using computer-aided techniques[2] to calculate the optimum values of the ω_i. These are beyond the scope of the present discussion. Only one class of this filter type, the *inverse Chebyshev filter*, to be discussed later, can be designed using straightforward analytical methods.